High-order perturbative expansions of multi-parameter Φ^4 quantum field theories

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Abstract

We present high-order pertubative expansions of multi-parameter Φ^4 quantum field theories with an N-component fundamental field, containing up to 4th-order polynomials of the field. Multi-parameter Φ^4 theories generalize the simplest $O(N)$-symmetric Φ^4 theories, and describe more complicated symmetry breaking patterns. These notes collect several high-order perturbative series of physically interesting multi-parameter Φ^4 theories, to five or six loops. We consider the O$(M)\otimes$O(N)-symmetric Φ^4 model, the so-called MN model, and a spin-density-wave Φ^4 model containing five quartic terms. The corresponding Tables of the coefficients are reported in Ref. [1].

I. INTRODUCTION

In the framework of the renormalization-group (RG) approach to critical phenomena, a quantitative description of many continuous phase transitions can be obtained by considering an effective Landau-Ginzburg-Wilson (LGW) Φ^4 field theory, containing up to fourth-order powers of the field components. The simplest example is the O(N)-symmetric Φ^4 theory, defined by the Lagrangian density

$$\mathcal{L}_{O(N)} = \frac{1}{2}\sum_i (\partial_\mu \Phi_i)^2 + \frac{1}{2} r \sum_i \Phi_i^2 + \frac{1}{4!} u (\sum_i \Phi_i^2)^2 \tag{1.1}$$

where Φ is an N-component real field. These Φ^4 theories describe are characterized by the symmetry breaking O(N)→O($N-1$). See, e.g., Refs. [2,3] for recent reviews discussing these models. Beside the transitions described by O(N) models, there are also other physically interesting transitions described by more general Landau-Ginzburg-Wilson (LGW) Φ^4 field theories, characterized by more complex symmetries and symmetry breaking patterns. The general LGW Φ^4 theory for an N-component field Φ_i can be written as

$$\mathcal{L} = \frac{1}{2}\sum_i (\partial_\mu \Phi_i)^2 + \frac{1}{2}\sum_i r_i \Phi_i^2 + \frac{1}{4!}\sum_{ijkl} u_{ijkl}\; \Phi_i \Phi_j \Phi_k \Phi_l \tag{1.2}$$

where the number of independent parameters r_i and u_{ijkl} depends on the symmetry group of the theory. Here, we are only assuming a parity symmetry which forbids third-order terms. An interesting class of models are those in which $\sum_i \Phi_i^2$ is the unique quadratic polynomial invariant under the symmetry group of the theory, corresponding to the case all field components become critical simultaneously. This requires that all r_i are equal, $r_i = r$, and u_{ijkl} must be such not to generate other quadratic invariant terms under RG transformations, for example, it must satisfy the trace condition [2] $\sum_i u_{iikl} \propto \delta_{kl}$. In these models, criticality is driven by tuning the single parameter r, which physically may correspond to the reduced temperature.

More general LGW Φ^4 theories, which allow for the presence of independent quadratic parameters r_i, must be considered to describe multicritical behaviors where there are independent correlation lengths that diverge simultaneously, which may arise from the competition of distinct types of ordering, see e.g. Ref. [4] and references therein. Note that, like the simplest O(N) models, all multi-parameter Φ^4 field theories are expected to be trivial in four dimensions.

High-order perturbative expansions, to five and six loops, of several multi-parameter Φ^4 theories have been computed in Refs. [3–20]. These notes collect several high-order series of physically interesting multi-parameter Φ^4 theories, to five or six loops. The corresponding Tables of the coefficients are reported in Ref. [1]. We consider two perturbative schemes: the three-dimensional (3D) massive zero-momentum (MZM) scheme in three dimensions and the massless (critical) $\overline{\rm MS}$ scheme. In the MZM scheme, one expands in powers of the MZM quartic couplings g_{ijkl}, defined by

$$\Gamma^{(2)}_{ij}(p) = \delta_{ij} Z_\phi^{-1} \left[m^2 + p^2 + O(p^4) \right], \qquad \Gamma^{(4)}_{ijkl}(0) = m\, Z_\phi^{-2}\, g_{ijkl} \tag{1.3}$$

The $\overline{\rm MS}$ scheme is based on a minimal subtraction procedure within the dimensional regularization, and can give rise to an $\epsilon \equiv 4-d$ expansion, and also 3D expansions in the renormalized $\overline{\rm MS}$ couplings g_{ijkl} by setting $\epsilon = 1$ after renormalization. The RG flow is determined by the FPs, which are common zeroes g^*_{ijkl} of the β-functions, $\beta_{ijkl}(g_{abcd}) \equiv m\partial g_{ijkl}/\partial m$ and $\beta_{ijkl}(g_{abcd}) \equiv \mu\partial g_{ijkl}/\partial \mu$ in the MZM and $\overline{\rm MS}$ schemes respectively. We report series for the O(M)⊗O(N)-symmetric Φ^4 model, the so-called MN model, and a spin-density-wave Φ^4 model containing five quartic terms. We also mention that high-order perturbative series for $U(N) \times U(N)$, SU(4)⊗SU(4), U(N) and SU(N) Φ^4 field theories have been computed in Refs. [13,16,19].

II. THE O(M)⊗O(N)-SYMMETRIC MODEL

The O(M)⊗O(N)-symmetric Φ^4 model is defined by the Hamiltonian density

$$\frac{1}{2}\sum_{ai}\Big[\sum_\mu(\partial_\mu\Phi_{ai})^2 + r\Phi_{ai}^2\Big] + \frac{u_0}{4!}\Big(\sum_{ai}\Phi_{ai}^2\Big)^2 + \frac{v_0}{4!}\Big[\sum_{i,j}\Big(\sum_a\Phi_{ai}\Phi_{aj}\Big)^2 - \Big(\sum_{ai}\Phi_{ai}^2\Big)^2\Big], \quad (2.1)$$

where Φ_{ai} is a real $N \times M$ matrix field ($a = 1, \dots, N$ and $i = 1, \dots, M$).

We also consider the four independent quadratic perturbations $Q^{(k)}$ that break the O(M)⊗O(N) symmetry, i.e.

$$\begin{aligned}
Q^{(1)}_{aibj} &= \Phi_{ai}\Phi_{bj} - \Phi_{aj}\Phi_{bi}, \qquad (2.2)\\
Q^{(2)}_{aibj} &= \tfrac{1}{2}\left(\Phi_{ai}\Phi_{bj} + \Phi_{aj}\Phi_{bi}\right) - \tfrac{1}{M}\delta_{ab}\Phi_{ci}\Phi_{cj} - \tfrac{1}{N}\delta_{ij}\Phi_{ak}\Phi_{bk} + \tfrac{1}{MN}\delta_{ab}\delta_{ij}\Phi_{ck}\Phi_{ck},\\
Q^{(3)}_{ij} &= \Phi_{ci}\Phi_{cj} - \tfrac{1}{N}\delta_{ij}\Phi_{ck}\Phi_{ck},\\
Q^{(4)}_{ab} &= \Phi_{ak}\Phi_{bk} - \tfrac{1}{M}\delta_{ab}\Phi_{ck}\Phi_{ck}.
\end{aligned}$$

The above four perturbations are related to different representations of the O(M) and O(N) groups. Therefore, they do no mix under renormalization-group (RG) transformations.

In the following we report the perturbative expansions in the massive zero-momentum (MZM) scheme and in the minimal-subtraction ($\overline{\rm MS}$) scheme. For further details see Refs. [3,9–12,14,15,17].

A. The 3D massive zero-momentum perturbative expansion

In the MZM scheme the theory is renormalized by introducing a set of zero-momentum conditions for the one-particle irreducible two-point and four-point correlation functions:

$$\Gamma^{(2)}_{ai,bj}(p) = \delta_{ai,bj} Z_\phi^{-1}\left[m^2 + p^2 + O(p^4)\right], \quad (2.3)$$

where $\delta_{ai,bj} \equiv \delta_{ab}\delta_{ij}$,

$$\Gamma^{(4)}_{ai,bj,ck,dl}(0) = Z_\phi^{-2} m \left(uS_{ai,bj,ck,dl} + vC_{ai,bj,ck,dl}\right), \quad (2.4)$$

and S, C are appropriate tensorial factors associated with the two quartic terms of Hamiltonian (2.1):

$$S_{ai,bj,ck,dl} \equiv \tfrac{1}{3}(\delta_{ai,bj}\delta_{ck,dl} + \delta_{ai,ck}\delta_{bj,dl} + \delta_{ai,dl}\delta_{bj,ck}), \tag{2.5}$$
$$C_{ai,bj,ck,dl} \equiv \tfrac{1}{6}\left[\delta_{ab}\delta_{cd}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \delta_{ac}\delta_{bd}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) + \delta_{ad}\delta_{bc}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl})\right] - S_{ai,bj,ck,dl}$$

Eqs. (2.3) and (2.4) relate the mass scale (inverse correlation length) m, and the zero-momentum quartic couplings u and v to the corresponding Hamiltonian parameters r, u_0, and v_0. In addition, the function Z_t is defined by the relation

$$\Gamma^{(1,2)}_{ai,bj}(0) = \delta_{ai,bj} Z_t^{-1}, \tag{2.6}$$

where $\Gamma^{(1,2)}$ is the one-particle irreducible two-point function with an insertion of $\frac{1}{2}\Phi^2$. The RG flow in the quartic-coupling u, v plane is determined by the β-functions

$$\beta_u(u,v) = m \left.\frac{\partial u}{\partial m}\right|_{u_0,v_0}, \qquad \beta_v(u,v) = m \left.\frac{\partial v}{\partial m}\right|_{u_0,v_0}. \tag{2.7}$$

Their common zeroes are the fixed points (FP's) of the RG flow. The RG functions η_ϕ and η_t associated with the standard critical exponents are

$$\eta_{\phi,t}(u,v) = \frac{\partial \ln Z_{\phi,t}}{\partial \ln m} = \beta_u \frac{\partial \ln Z_{\phi,t}}{\partial u} + \beta_v \frac{\partial \ln Z_{\phi,t}}{\partial v}, \tag{2.8}$$

In order to compute the RG dimensions of the quadratic operators (2.2), one computes the four functions Z_{Qk} defined through the relation $\Gamma^{(Qk,2)}_{ai,bj}(0) = q_{k,ai,bj} Z_{Qk}^{-1}$, where $\Gamma^{(Qk,2)}$ is the one-particle irreducible two-point function with an insertion of $Q^{(k)}$ ($q_{k,ai,bj}$ are the appropriate tensorial factors). One then derives the corresponding RG functions η_{Qk} through the equation

$$\eta_{Qk}(u,v) = \frac{\partial \ln Z_{Qk}}{\partial \ln m} \tag{2.9}$$

The critical exponents η and ν and the RG dimensions y_{Qk} of the quadratic operators $Q^{(k)}$ are obtained by evaluating the RG functions at the fixed point u^*, v^*, as

$$\begin{aligned} \eta &= \eta_\phi(u^*,v^*), \\ \nu &= \left[2 - \eta_\phi(u^*,v^*) + \eta_t(u^*,v^*)\right]^{-1}, \\ y_{Qk} &= 2 - \eta_\phi(u^*,v^*) + \eta_{Qk}(u^*,v^*). \end{aligned} \tag{2.10}$$

The 3D expansions of the RG functions in powers of the zero-momentum couplings u, v are [we set $A = 3/(16\pi)$]

$$\begin{aligned} \beta_u = \sum_{ijkl} b^{(u)}_{ijkl} M^k N^l u^i v^j = & -u + \tfrac{8+MN}{9} A u^2 - \tfrac{2(1-M)(1-N)}{9} A uv + \tfrac{(1-M)(1-N)}{9} A v^2 \\ & - \tfrac{760+164MN}{2187} A^2 u^3 + \tfrac{400(1-M)(1-N)}{2187} A^2 u^2 v - \tfrac{118(1-M)(1-N)}{729} A^2 u v^2 \\ & + \tfrac{90(1-M)(1-N)}{2187} A^2 v^3 + ... \end{aligned} \tag{2.11}$$

$$\beta_v = \sum_{ijkl} b^{(v)}_{ijkl} M^k N^l u^i v^j = -v - \tfrac{8-M-N}{9} A v^2 + \tfrac{4}{3} A u v - \tfrac{1480+92MN}{2187} A^2 u^2 v$$
$$+ \tfrac{1912-400M-400N+184MN}{2187} A^2 u v^2 - \tfrac{298-106M-106N+40MN}{729} A^2 v^3 + ...$$
$$\eta_\phi = \sum_{ijkl} e^{(\phi)}_{ijkl} M^k N^l u^i v^j = \tfrac{16+8MN}{2187} A^2 u^2 - \tfrac{16(1-M)(1-N)}{2187} A^2 uv + \tfrac{4(1-M)(1-N)}{729} A^2 v^2 + ...$$
$$\eta_t = \sum_{ijkl} e^{(t)}_{ijkl} M^k N^l u^i v^j = -\tfrac{2+MN}{9} A u + \tfrac{(1-M)(1-N)}{9} A v + \tfrac{4+2MN}{81} A^2 u^2 - \tfrac{4(1-M)(1-N)}{81} A^2 uv$$
$$+ \tfrac{(1-M)(1-N)}{27} A^2 v^2 + ...$$
$$\eta_{Q1} = \sum_{ijkl} e^{(Q1)}_{ijkl} M^k N^l u^i v^j = -\tfrac{2}{9} A u + \tfrac{1}{9} A v + \tfrac{12+2MN}{243} A^2 u^2 - \tfrac{28-4M-4N+4MN}{243} A^2 uv$$
$$+ \tfrac{17-5M-5N+2MN}{243} A^2 v^2 + ...$$
$$\eta_{Q2} = \sum_{ijkl} e^{(Q2)}_{ijkl} M^k N^l u^i v^j = -\tfrac{2}{9} A u + \tfrac{1}{9} A v + \tfrac{12+2MN}{243} A^2 u^2 - \tfrac{12-4M-4N+4MN}{243} A^2 uv$$
$$+ \tfrac{9-3M-3N+2MN}{243} A^2 v^2 + ...$$
$$\eta_{Q3} = \sum_{ijkl} e^{(Q3)}_{ijkl} M^k N^l u^i v^j = -\tfrac{2}{9} A u + \tfrac{1-N}{9} A v + \tfrac{12+2MN}{243} A^2 u^2 - \tfrac{12-4M-12N+4MN}{243} A^2 uv$$
$$+ \tfrac{3-M-3N+MN}{81} A^2 v^2 + ...$$
$$\eta_{Q4} = \sum_{ijkl} e^{(Q4)}_{ijkl} M^k N^l u^i v^j = -\tfrac{2}{9} A u + \tfrac{1-M}{9} A v + \tfrac{12+2MN}{243} A^2 u^2 - \tfrac{12-12M-4N+4MN}{243} A^2 uv$$
$$+ \tfrac{3-3M-N+MN}{81} A^2 v^2 + ...$$

The values of the coefficients $b^{(u)}_{ijkl}$, $b^{(v)}_{ijkl}$, $e^{(\phi)}_{ijkl}$, $e^{(t)}_{ijkl}$, and $e^{(Qk)}_{ijkl}$ up to six loops are reported in the file `OMN-MZM.TXT` attached to Ref. [1]. Each line of this file contains 7 numbers. The first one indicates the quantity at hand: 1,2,3,4,5,6,7 correspond respectively to β_u, β_v, η_ϕ, η_t, η_{Q1}, η_{Q2}, η_{Q3} (the expansion of η_{Q4} can be obtained from that of η_{Q3} by interchanging M with N); the second integer number gives the number of loops; the subsequent four integer numbers are the indices i, j, k, l; finally, the last real number is the value of the coefficient.

B. The minimal-subtraction pertubative expansion

In the $\overline{\rm MS}$ scheme one sets

$$\Phi = [Z_\phi(u,v)]^{1/2} \Phi_R, \qquad (2.12)$$
$$u_0 = A_d \mu^\epsilon Z_u(u,v),$$
$$v_0 = A_d \mu^\epsilon Z_v(u,v),$$

where the renormalization functions Z_ϕ, Z_u, and Z_v are determined from the divergent part of the two- and four-point one-particle irreducible correlation functions computed in dimensional regularization. They are normalized so that $Z_\phi(u,v) \approx 1$, $Z_u(u,v) \approx u$, and $Z_v(u,v) \approx v$ at tree level. Here A_d is a d-dependent constant given by $A_d \equiv 2^{d-1}\pi^{d/2}\Gamma(d/2)$. Moreover, one defines a mass renormalization constant $Z_t(u,v)$ by requiring $Z_t \Gamma^{(1,2)}$ to be

finite when expressed in terms of u and v. Here $\Gamma^{(1,2)}$ is the one-particle irreducible two-point function with an insertion of $\frac{1}{2}\Phi^2$. The β functions,

$$\beta_u(u,v) = \mu \left.\frac{\partial u}{\partial \mu}\right|_{u_0,v_0}, \qquad \beta_v(u,v) = \mu \left.\frac{\partial v}{\partial \mu}\right|_{u_0,v_0}, \tag{2.13}$$

have a simple dependence on d:

$$\beta_u = (d-4)u + B_u(u,v), \qquad \beta_v = (d-4)v + B_v(u,v), \tag{2.14}$$

where the functions $B_u(u,v)$ and $B_v(u,v)$ are independent of d. The RG dimensions of the quadratic operators $Q^{(k)}$ are obtained by computing the renormalization functions $Z_{Qk}(u,v)$. These functions are determined by requiring $Z_{Qk}\Gamma^{(Qk,2)}$ to be finite when expressed in terms of u and v. Here $\Gamma^{(Qk,2)}$ is the one-particle irreducible two-point function with an insertion of $Q^{(k)}$. The RG functions η_ϕ and η_t, associated with the critical exponents, are obtained from

$$\eta_{\phi,t}(u,v) = \left.\frac{\partial \log Z_{\phi,t}}{\partial \log \mu}\right|_{u_0,v_0}. \tag{2.15}$$

The same equation allows the determination of η_{Qk} from Z_{Qk}. The RG functions η_ϕ, η_t, and η_{Qk} are independent of d. The standard critical exponents η and ν, and the RG dimensions y_{Qk} of the quadratic perturbations Q_k are obtained by using Eq. (2.10).

The expansions of the RG functions in powers of the $\overline{\rm MS}$ couplings u, v are

$$\begin{aligned}
B_u &= \sum_{ijkl} b^{(u)}_{ijkl} M^k N^l u^i v^j = \tfrac{8+MN}{6}u^2 - \tfrac{(1-M)(1-N)}{3}uv + \tfrac{(1-M)(1-N)}{6}v^2 - \tfrac{14+3MN}{12}u^3 \\
&\quad + \tfrac{11(1-M)(1-N)}{18}u^2 v - \tfrac{13(1-M)(1-N)}{24}uv^2 + \tfrac{5(1-M)(1-N)}{36}v^3 + \ldots \\
B_v &= \sum_{ijkl} b^{(v)}_{ijkl} M^k N^l u^i v^j = 2uv - \tfrac{8-M-N}{6}v^2 - \tfrac{82+5MN}{36}u^2 v + \tfrac{53-11M-11N+5MN}{18}uv^2 \\
&\quad - \tfrac{99-35M-35N+13MN}{72}v^3 + \ldots \\
\eta_\phi &= \sum_{ijkl} e^{(\phi)}_{ijkl} M^k N^l u^i v^j = \tfrac{2+MN}{72}u^2 - \tfrac{(1-M)(1-N)}{36}uv + \tfrac{(1-M)(1-N)}{48}v^2 + \ldots \\
\eta_t &= \sum_{ijkl} e^{(t)}_{ijkl} M^k N^l u^i v^j = -\tfrac{2+MN}{6}u + \tfrac{(1-M)(1-N)}{6}v \\
&\quad + \tfrac{2+MN}{12}u^2 - \tfrac{(1-M)(1-N)}{6}uv + \tfrac{(1-M)(1-N)}{8}v^2 + \ldots \\
\eta_{Q1} &= \sum_{ijkl} e^{(Q1)}_{ijkl} M^k N^l u^i v^j = -\tfrac{1}{3}u + \tfrac{1}{2}v + \tfrac{6+MN}{36}u^2 - \tfrac{7-M-N+MN}{18}uv + \tfrac{17-5M-5N+2MN}{72}v^2 + \ldots \\
\eta_{Q2} &= \sum_{ijkl} e^{(Q2)}_{ijkl} M^k N^l u^i v^j = -\tfrac{1}{3}u + \tfrac{1}{6}v + \tfrac{6+MN}{36}u^2 - \tfrac{3-M-N+MN}{18}uv + \tfrac{9-3M-3N+2MN}{72}v^2 + \ldots \\
\eta_{Q3} &= \sum_{ijkl} e^{(Q3)}_{ijkl} M^k N^l u^i v^j = -\tfrac{1}{3}u + \tfrac{1-N}{6}v + \tfrac{6+MN}{36}u^2 - \tfrac{3-M-3N+MN}{18}uv + \tfrac{3-M-3N+MN}{24}v^2 + \ldots \\
\eta_{Q4} &= \sum_{ijkl} e^{(Q4)}_{ijkl} M^k N^l u^i v^j = -\tfrac{1}{3}u + \tfrac{1-M}{6}v + \tfrac{6+MN}{36}u^2 - \tfrac{3-3M-N+MN}{18}uv + \tfrac{3-3M-N+MN}{24}v^2 + \ldots
\end{aligned} \tag{2.16}$$

The values of the coefficients $b^{(u)}_{ijkl}$, $b^{(v)}_{ijkl}$, $e^{(\phi)}_{ijkl}$, $e^{(t)}_{ijkl}$, and $e^{(Qk)}_{ijkl}$ up to five loops are reported in the file `OMN-MS.TXT` attached to Ref. [1]. The meaning of the numbers reported in this file is the same as that of the numbers appearing in file `OMN-MZM.TXT`, see the end of Sec. II A. In file `OMN-MS.TXT` the coefficients are given numerically for simplicity, although we computed them exactly in terms of fractions and ζ functions.

III. THE MN MODEL

The so-called mn model is defined by the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\sum_{ai}\Big[\sum_{\mu}(\partial_\mu\Phi_{ai})^2 + r\Phi_{ai}^2\Big] + \frac{u_0}{4!}\Big(\sum_{ai}\Phi_{ai}^2\Big)^2 + \frac{v_0}{4!}\sum_{abi}\Phi_{ai}^2\Phi_{bi}^2, \tag{3.1}$$

where Φ_{ai} is a real $m\times n$ matrix, i.e., $a = 1,\dots,m$ and $i = 1,\dots,n$.

We refer to Refs. [3,7,8,18] for further details on the perturbative expansions in the mn model.

A. The 3D massive zero-momentum perturbative expansion

The basic relations in the MZM scheme are the same as those reported in Sec. II A. Beside Eq. (2.3), we have

$$\Gamma^{(4)}_{ai,bj,ck,dl}(0) = Z_\phi^{-2} m\left(uS_{ai,bj,ck,dl} + vC_{ai,bj,ck,dl}\right) \tag{3.2}$$

where S and C are the tensorial factors corresponding to the quartic terms of the mn Hamiltonian, i.e.

$$\begin{aligned} S_{ai,bj,ck,dl} &= \tfrac{1}{3}(\delta_{ai,bj}\delta_{ck,dl} + \delta_{ai,ck}\delta_{bj,dl} + \delta_{ai,dl}\delta_{bj,ck}),\\ C_{ai,bj,ck,dl} &= \delta_{ij}\delta_{ik}\delta_{il}\,\tfrac{1}{3}\left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right). \end{aligned} \tag{3.3}$$

The function Z_t is defined as in Eq. (2.6). The β-functions $\beta_{u,v}$ and RG functions $\eta_{\phi,t}$ are defined as in Eqs. (2.7) and (2.8). Their expansions are [we set $A = 3/(16\pi)$]:

$$\begin{aligned} \beta_u &= \sum_{ijkl} b^{(u)}_{ijkl} m^k n^l u^i v^j = -u + \tfrac{8+mn}{9}Au^2 + \tfrac{4+2m}{9}Auv - \tfrac{760+164mn}{2187}A^2u^3 \\ &\qquad - \tfrac{800+400m}{2187}A^2u^2v - \tfrac{184+92m}{2187}A^2uv^2 + \dots \\ \beta_v &= \sum_{ijkl} b^{(v)}_{ijkl} m^k n^l u^i v^j = -v + \tfrac{8+m}{9}Av^2 + \tfrac{4}{3}Auv - \tfrac{1480+92mn}{2187}A^2u^2v - \tfrac{2096+400m}{2187}A^2uv^2 \\ &\qquad - \tfrac{760+164m}{2187}A^2v^3 + \dots \\ \eta_\phi &= \sum_{ijkl} e^{(\phi)}_{ijkl} m^k n^l u^i v^j = \tfrac{16+8mn}{2187}A^2u^2 + \tfrac{32+16m}{2187}A^2uv + \tfrac{16+8m}{2187}A^2v^2 + \dots \\ \eta_t &= \sum_{ijkl} e^{(t)}_{ijkl} m^k n^l u^i v^j = -\tfrac{2+mn}{9}Au - \tfrac{2+m}{9}Av + \tfrac{4+2mn}{81}A^2u^2 + \tfrac{8+4m}{81}A^2uv + \tfrac{4+2m}{81}A^2v^2 + \dots \end{aligned} \tag{3.4}$$

The values of the coefficients $b^{(u)}_{ijkl}$, $b^{(v)}_{ijkl}$, $e^{(\phi)}_{ijkl}$, and $e^{(t)}_{ijkl}$ up to six loops are reported in the file `MN-MZM.TXT` attached to Ref. [1]. Each line of this file contains 7 numbers. The first one indicates the quantity at hand: 1,2,3,4 correspond respectively to β_u, β_v, η_ϕ, and η_t; the second integer number gives the number of loops; the subsequent four integer numbers are the indices i, j, k, l; finally, the last real number is the value of the coefficient.

B. The minimal-subtraction perturbative expansion

The RG functions in the $\overline{\rm MS}$ scheme are

$$\begin{aligned}
B_u &= \sum_{ijkl} b^{(u)}_{ijkl} m^k n^l u^i v^j = \tfrac{8+mn}{6}u^2 + \tfrac{2+m}{3}uv - \tfrac{14+3mn}{12}u^3 - \tfrac{22+11m}{18}u^2 v - \tfrac{10+5m}{36}uv^2 + ... \qquad (3.5)\\
B_v &= \sum_{ijkl} b^{(v)}_{ijkl} m^k n^l u^i v^j = 2uv + \tfrac{8+m}{6}v^2 - \tfrac{82+5mn}{36}u^2 v - \tfrac{58+11m}{18}uv^2 - \tfrac{14+3m}{12}v^3 + ...\\
\eta_\phi &= \sum_{ijkl} e^{(\phi)}_{ijkl} m^k n^l u^i v^j = \tfrac{2+mn}{72}u^2 + \tfrac{2+m}{36}uv + \tfrac{2+m}{72}v^2 + ...\\
\eta_t &= \sum_{ijkl} e^{(t)}_{ijkl} m^k n^l u^i v^j = -\tfrac{2+mn}{6}u - \tfrac{2+m}{6}v + \tfrac{2+mn}{12}u^2 + \tfrac{2+m}{6}uv + \tfrac{2+m}{12}v^2 + ...
\end{aligned}$$

The values of the coefficients $b^{(u)}_{ijkl}$, $b^{(v)}_{ijkl}$, $e^{(\phi)}_{ijkl}$, and $e^{(t)}_{ijkl}$ up to five loops are reported in the file `MN-MS.TXT` attached to Ref. [1]. The meaning of the numbers reported in this file is the same as that of the numbers appearing in file `MN-MZM.TXT`, see the end of Sec. III A.

IV. THE SPIN-DENSITY-WAVE Φ^4 MODEL

The spin-density-wave (SDW) Φ^4 model is defined by the Hamiltonian density

$$\begin{aligned}
\mathcal{H} = &|\partial_\mu \Phi_1|^2 + |\partial_\mu \Phi_2|^2 + r(|\Phi_1|^2 + |\Phi_2|^2) + \frac{u_{1,0}}{2}(|\Phi_1|^4 + |\Phi_2|^4) + \\
&+ \frac{u_{2,0}}{2}(|\Phi_1^2|^2 + |\Phi_2^2|^2) + w_{1,0}|\Phi_1|^2|\Phi_2|^2 + w_{2,0}|\Phi_1\Phi_2|^2 + w_{3,0}|\Phi_1^*\Phi_2|^2 \qquad (4.1)
\end{aligned}$$

where Φ_{ai} is a complex $2 \times N$ matrix field ($a = 1, 2$ and $i = 1, ...N$).

We refer to Refs. [20] for further details on the perturbative expansions in this model.

A. The 3D massive zero-momentum perturbative expansion

The RG functions of the SDW model in the MZM scheme are defined following the same steps as in the cases considered in the preceding sections, see Sec. II A.

The β-functions of the renormalized quartic couplings u_i, w_i corresponding to the quartic Hamiltoninan parameters $u_{i,0}$, $w_{i,0}$ can be written as

$$\beta_\# = \sum_{ijklmp} b^{(\#)}_{ijklmp} N^p u_1^i u_2^j w_1^k w_3^l w_3^m \qquad (4.2)$$

where the symbol # indicates u_1, u_2, w_1, w_2, w_3. At one loop we have ($A \equiv 1/(8\pi)$)

$$\begin{aligned}
\beta_{u_1} &= -u_1 + A\left[(N+4)u_1^2 + 4u_1u_2 + 4u_2^2 + Nw_1^2 + w_2^2 + w_3^2 + 2w_1w_2 + 2w_1w_3\right] + \dots \\
\beta_{u_2} &= -u_2 + A\left[6u_1u_2 + Nu_2^2 + 2w_2w_3\right] + \dots \\
\beta_{w_1} &= -w_1 + A\left[2w_1^2 + w_2^2 + w_3^2 + 2(N+1)u_1w_1 + 4u_2w_1 + 2u_1w_2 + 2u_1w_3\right] + \dots \\
\beta_{w_2} &= -w_2 + A\left[Nw_2^2 + 2u_1w_2 + 4u_2w_3 + 4w_1w_2 + 2w_2w_3\right] + \dots \\
\beta_{w_3} &= -w_3 + A\left[Nw_3^2 + 2u_1w_3 + 4u_2w_2 + 4w_1w_3 + 2w_2w_3\right] + \dots
\end{aligned}$$

Analogous expansions apply to $e^{(\phi)}$ and $e^{(t)}$ with coefficients $e^{(\phi)}_{ijklmp}$, and $e^{(t)}_{ijklmp}$. The values of the coefficients $b^{(\#)}_{ijklmp}$, $e^{(\phi)}_{ijklmp}$, and $e^{(t)}_{ijklmp}$ up to six loops are reported in the file `SDW-MZM.TXT` attached to Ref. [1]. The file is formatted as before. Nine numbers appear in each line. The first one indicates the quantity one is considering: 1,2,3,4,5 correspond to the β functions β_{u_1}, β_{u_2}, β_{w_1}, β_{w_2}, and β_{w_3}; 6 and 7 to $e^{(\phi)}$ and $e^{(t)}$. The second number gives the number of loops. The subsequent six integer numbers correspond to i, j, k, l, m, p. Finally, the last number gives the coefficient.

B. The minimal-subtraction perturbative expansion

The RG functions of the SDW model in the $\overline{\rm MS}$ scheme are defined following the same steps as in the cases considered in the preceding sections, see Sec. II B.

The β-functions of the renormalized quartic couplings u_i, w_i corresponding to the quartic Hamiltoninan parameters $u_{i,0}$, $w_{i,0}$ can be written as

$$\beta_\# = (d-4)\# + B_\#, \qquad B_\# = \sum_{ijklmp} b^{(\#)}_{ijklmp} N^p u_1^i u_2^j w_1^k w_3^l w_3^m \tag{4.3}$$

where the symbol # represents u_1, u_2, w_1, w_2, w_3. At one loop we have

$$\begin{aligned}
B_{u_1} &= (N+4)u_1^2 + 4u_1u_2 + 4u_2^2 + Nw_1^2 + w_2^2 + w_3^2 + 2w_1w_2 + 2w_1w_3 \\
B_{u_2} &= 6u_1u_2 + Nu_2^2 + 2w_2w_3 \\
B_{w_1} &= 2w_1^2 + w_2^2 + w_3^2 + 2(N+1)u_1w_1 + 4u_2w_1 + 2u_1w_2 + 2u_1w_3 \\
B_{w_2} &= Nw_2^2 + 2u_1w_2 + 4u_2w_3 + 4w_1w_2 + 2w_2w_3 \\
B_{w_3} &= Nw_3^2 + 2u_1w_3 + 4u_2w_2 + 4w_1w_3 + 2w_2w_3
\end{aligned} \tag{4.4}$$

The functions $e^{(\phi)}$ and $e^{(t)}$ have analogous expansions with coefficients $e^{(\phi)}_{ijklmp}$, and $e^{(t)}_{ijklmp}$. The values of the coefficients $b^{(\#)}_{ijklmp}$, $e^{(\phi)}_{ijklmp}$, and $e^{(t)}_{ijklmp}$ up to five loops are reported in the file `SDW-MS.TXT` attached to Ref. [1]. The meaning of the numbers reported in this file is the same as that of the numbers appearing in file `SDW-MZM.TXT`, see the end of Sec. IV A.

C. Perturbations of the O(4)⊗O(N) fixed points

We report the perturbative series of the RG eigenvalues Ω_1 and Ω_2 defined in App. B of the paper. We write

$$\Omega_{\#} = \sum_{ijk} \Omega^{\#}_{ijk} N^k u_1^i u_2^j. \tag{4.5}$$

The values of the coefficients $\Omega^{\#}_{ijk}$ are reported in the file `Perturbations-O4ON.TXT` attached to Ref. [1]. In each line we report 6 numbers. The first one specifies the quantity one is referring to: 1 corresponds to Ω_1 in the MZM scheme; 2 corresponds to Ω_2 in the MZM scheme; 3 corresponds to Ω_1 in the $\overline{\text{MS}}$ scheme; 4 corresponds to Ω_2 in the $\overline{\text{MS}}$ scheme. The second number gives the number of loops, the third, fourth, and fifth number refer to the indices i, j, and k appearing in Eq. (4.5). Finally, the last real number is the coefficient $\Omega^{\#}_{ijk}$. For $N = 3$ these series are reported in App. B.

D. Perturbations of the mn fixed points

We report the perturbative series of the RG eigenvalues Ω_1 and Ω_2 defined in App. C of the paper. We write

$$\Omega_{\#} = \sum_{ijk} \Omega^{\#}_{ijk} N^k u_1^i u_2^j \, . \tag{4.6}$$

The values of the coefficients $\Omega^{\#}_{ijk}$ are reported in the file `Perturbations-MN.TXT` attached to Ref. [1]. In each line we report 6 numbers. The first one specifies the quantity one is referring to: 1 corresponds to Ω_1 in the MZM scheme; 2 corresponds to Ω_2 in the MZM scheme; 3 corresponds to Ω_1 in the $\overline{\text{MS}}$ scheme; 4 corresponds to Ω_2 in the $\overline{\text{MS}}$ scheme. The second number gives the number of loops, the third, fourth, and fifth number refer to the indices i, j, and k appearing in Eq. 4.6. Finally, the last number is the coefficient $\Omega^{\#}_{ijk}$.

REFERENCES

[1] EPAPS Document, No. E-PRBMDO-74-057634, of Ref. [20]. This document can be reached via a direct link in the corresponding online article's HTML reference section or via the EPAPS homepage (http://www.aip.org/pubservs/epaps.html).

[2] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, (Clarendon Press, Oxford, 1989), fourth edition Oxford 2002.

[3] A. Pelissetto, E. Vicari, *Critical Phenomena and Renormalization Group Theory*, Phys. Rep. 368 (2002) 549 [`arXiv:cond-mat/0012164`].

[4] P. Calabrese, A. Pelissetto, E. Vicari, *Multicritical behavior of $O(n_1)\oplus O(n_2)$-symmetric systems*, Phys. Rev. B 67 (2003) 054505 [`arXiv:cond-mat/0209580`]

[5] H. Kleinert, V. Schulte-Frohlinde, *Exact five-loop renormalization group functions of ϕ^4-theory with $O(N)$-symmetric and cubic interactions. Critical exponents up to ϵ^5*, Phys. Lett. B 342 (1995) 284 [`arXiv:cond-mat/9503038`]

[6] D.V. Pakhnin, A.I. Sokolov, *Five-loop renormalization-group expansions for the three-dimensional n-vector cubic model and critical exponents for impure Ising systems*, Phys. Rev. B 61 (2000) 15130 [`arXiv:cond-mat/9912071`].

[7] J. M. Carmona, A. Pelissetto, E. Vicari, *The N-component Ginzburg-Landau Hamiltonian with cubic symmetry: a six-loop study*, Phys. Rev. B 61 (2000) 15136 [`arXiv:cond-mat/0002402`].

[8] A. Pelissetto, E. Vicari, *Randomly dilute spin models: a six-loop field-theoretic study*, Phys. Rev. B 62 (2000) 6393 [`arXiv:cond-mat/0002402`].

[9] A. Pelissetto, P. Rossi, E. Vicari, *The critical behavior of frustrated spin models with noncollinear order*, Phys. Rev. B 63 (2001) 140414(R) [`arXiv:cond-mat/0007389`]; *Chiral exponents in frustrated spin models with noncollinear order*, Phys. Rev. B 65 (2002) 020403(R) [`arXiv:cond-mat/0106525`].

[10] A. Pelissetto, P. Rossi, E. Vicari, *Large-N critical behavior of $O(M)\times O(N)$ spin models*, Nucl. Phys. B 607 (2001) 605 [`arXiv:hep-th/0104024`]

[11] P. Parruccini, *Critical behavior of frustrated spin systems with nonplanar orderings*, Phys. Rev. B 68 (2003) 104415 [`arXiv:cond-mat/0305287`].

[12] P. Calabrese, P. Parruccini, *Five-loop epsilon expansion for $O(n)\times O(m)$ spin models*, Nucl. Phys. B 679 (2004) 568 [`arXiv:cond-mat/0308037`].

[13] A. Butti, A. Pelissetto, E. Vicari, *On the nature of the finite-temperature chiral transition in QCD*, JHEP 08 (2003) 029 [`arXiv:hep-ph/0307036`]

[14] M. De Prato, A. Pelissetto, E. Vicari, *The normal-to-planar superfluid transition in ^{3}He*, Phys. Rev. B 70 (2004) 214519 [`arXiv:cond-mat/0312362`].

[15] P. Calabrese, P. Parruccini, A. Pelissetto, E. Vicari, *Crossover behavior in three-dimensional dilute Ising systems*, Phys. Rev. E 69 (2004) 036120 [`arXiv:cond-mat/0405667`].

[16] P. Calabrese, P. Parruccini, *Five-loop epsilon expansion for U(n)xU(m) models: finite-temperature phase transition in light QCD*, JHEP 05 (2004) 018 [`arXiv:hep-ph/0403140`].

[17] P. Calabrese, A. Pelissetto, E. Vicari, *Multicritical behavior in frustrated spin systems with noncollinear order*, Nucl. Phys. B 709 (2005) 550 [`arXiv:cond-mat/0408130`].

[18] A. Pelissetto, E. Vicari, *Interacting N-vector order parameters with $O(N)$ symmetry*, Cond. Matt. Phys. (Ukraine) 8 (2005) 87 [`arXive:hep-th/0409214`].
[19] F. Basile, A. Pelissetto, E. Vicari, *The finite-temperature chiral transition in QCD with adjoint fermions*, JHEP 02 (2005) 044 [`arXiv:hep-th/0412026`]
[20] M. De Prato, A. Pelissetto, E. Vicari, *Spin-density-wave order in cuprates*, Phys. Rev. B 74 (2006) 144507 [`arXive:cond-mat/0601404`].